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The SO(6, 2) model of SU(3) and its generalisation to SU(n)

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Abstract. The complementarity relation between the U(p, q) and U(n) Lie groups enables us to reformulate the SO(6, 2) model of SU(3) in terms of a U(1, 1) group and to outline its generalisation to a family of $n-2$ models of SU(n), for $n \geqslant 3$, respectively associated with U($n-2$, 1), U($n-1$, 2), ..., and U(1, $n-2$) groups.

1. Introduction

Following Bernšteĭn *et al* (1975), a model of a compact Lie group G is defined as a realisation of a representation of G which consists of a direct sum of irreducible representations (irreps), containing exactly one representative from every equivalence class of irreps of G. For SU(2), several models are known (Schwinger 1965, Biedenharn and Louck 1981, Bracken and MacGibbon 1984, Bracken 1984, Van der Jeugt 1985), out of which Schwinger's UW_2 model (1965) looks the most attractive, thanks to the use of boson calculus computational advantages, even if it is not semisimple.

Recently a remarkable model of SU(3) has been discovered independently by Biedenharn and Flath (1984) and Bracken and MacGibbon (1984). The model space, which is a subspace $\mathscr{B}$ of the six-boson Fock space $\mathscr{F}_6$, carries an irrep of an SO(6, 2) group. Biedenharn and Flath have emphasised that this model provides the framework for a global algebraic formulation of the SU(3) tensor operator structure, thereby leading to a complete resolution of the multiplicity problem for such operators. On the other hand, Bracken and MacGibbon have stressed that the whole representation space can be generated by applying some modified boson creation operators to a vacuum state, in much the same way as the basis vectors are constructed in the Schwinger UW_2 model of SU(2).

The purpose of the present paper consists in simplifying the Bracken and MacGibbon analysis for SU(3) and outlining its generalisation to SU(n), while using the results recently obtained by one of us (Quesne 1986) about the complementarity relation between the U(p, q) and U(n) Lie groups (Kashiwara and Vergne 1978).

In § 2, a family of $n-2$ models of SU(n) is defined. In § 3, bases of the corresponding model spaces are constructed. In § 4, the model of SU(3) so obtained is shown to coincide with the SO(6, 2) model. Finally, § 5 contains the conclusion.

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2. Models of SU(n)

Let η_{is} and $\xi_{is} = (\eta_{is})^\dagger$, where $i = 1, \ldots, n-1$ and $s = 1, \ldots, n$, denote $(n-1)n$ pairs of boson creation and annihilation operators. When acting on the boson vacuum state $|0\rangle$, the operators η_{is} generate an $(n-1)n$-boson Fock space $\mathscr{F}_{(n-1)n}$. Under the real symplectic group Sp$[2(n-1)n, R]$, whose generators are realised by the operators (Moshinsky and Quesne 1971)

$$\mathbb{D}^\dagger_{is,jt} = \eta_{is}\eta_{jt} \qquad \mathbb{D}_{is,jt} = \xi_{is}\xi_{jt} \qquad \mathbb{E}_{is,jt} = \tfrac{1}{2}(\eta_{is}\xi_{jt} + \xi_{jt}\eta_{is}) \tag{2.1}$$

$\mathscr{F}_{(n-1)n}$ decomposes into the direct sum of two subspaces, made of all the boson states with an even or odd boson number respectively. Either subspace carries a positive discrete series irrep of Sp$[2(n-1)n, R]$—or more exactly of the double covering group Mp$[2(n-1)n]$ of Sp$[2(n-1)n, R]$—characterised by its lowest weight $\langle(1/2)^{(n-1)n}\rangle$ or $\langle(1/2)^{(n-1)n}3/2\rangle$.

Let us consider the following group chain (Quesne 1986):

$$\mathrm{Sp}[2(n-1)n, R] \supset \mathrm{U}(pn, qn) \supset \mathrm{U}(p, q) \times \mathrm{U}(n) \tag{2.2}$$

where p and q are any two positive integers subject to the condition

$$p + q = n - 1. \tag{2.3}$$

The U(pn, qn) basis operators $\mathbb{P}_{is,jt}$, where $i, j = 1, \ldots, n-1$ and $s, t = 1, \ldots, n$ are defined by

$$\begin{aligned} \mathbb{P}_{is,jt} &= \mathbb{E}_{is,jt} && \text{if } i, j = 1, \ldots, p \\ &= \mathbb{E}_{jt,is} && \text{if } i, j = p+1, \ldots, n-1 \\ &= \mathbb{D}^\dagger_{is,jt} && \text{if } i = 1, \ldots, p \text{ and } j = p+1, \ldots, n-1 \\ &= \mathbb{D}_{is,jt} && \text{if } i = p+1, \ldots, n-1 \text{ and } j = 1, \ldots, p \end{aligned} \tag{2.4}$$

and the U(p, q) and U(n) ones are obtained from them by contraction over the index s or i as follows:

$$P_{ij} = \sum_s \mathbb{P}_{is,js} \qquad i, j = 1, \ldots, n-1 \tag{2.5}$$

$$\mathscr{P}_{st} = \sum_i \varepsilon_i \mathbb{P}_{is,it} \qquad s, t = 1, \ldots, n \tag{2.6}$$

where

$$\begin{aligned} \varepsilon_i &= +1 && \text{if } i = 1, \ldots, p \\ &= -1 && \text{if } i = p+1, \ldots, n-1. \end{aligned} \tag{2.7}$$

The U(p, q) basis operators can also be written as

$$\begin{aligned} P_{ij} &= E_{ij} && \text{if } i, j = 1, \ldots, p \\ &= E_{ji} && \text{if } i, j = p+1, \ldots, n-1 \\ &= D^\dagger_{ij} && \text{if } i = 1, \ldots, p \text{ and } j = p+1, \ldots, n-1 \\ &= D_{ij} && \text{if } i = p+1, \ldots, n-1 \text{ and } j = 1, \ldots, p \end{aligned} \tag{2.8}$$

where $D^\dagger_{ij}$, D_{ij} and E_{ij} denote the contractions over s of the operators (2.1).

The U(pn, qn) irreps $[\rho]$ contained in either irrep of Sp$[2(n-1)n, R]$ (Kashiwara and Vergne 1978, King and Wybourne 1985, Quesne 1986) are positive discrete series irreps characterised by a single integer $\rho \in Z$, related to the eigenvalue $\rho + (p-q)n/2$ of the U(pn, qn) first-order Casimir operator

$$\mathbb{G}_1 = \sum_{is} \varepsilon_i \mathbb{P}_{is,is}. \tag{2.9}$$

The branching rules are given by

$$\langle (1/2)^{(n-1)n} \rangle \downarrow \sum_{\substack{\rho=-\infty \\ \rho \text{ even}}}^{+\infty} \oplus [\rho]$$

and

$$\langle (1/2)^{(n-1)n-1} 3/2 \rangle \downarrow \sum_{\substack{\rho=-\infty \\ \rho \text{ odd}}}^{+\infty} \oplus [\rho]. \tag{2.10}$$

The U(p, q) and U(n) groups are complementary (Moshinsky and Quesne 1970) within any irrep $[\rho]$ of U(pn, qn), meaning that the U(p, q) $\times$ U(n) irreps contained in $[\rho]$ are multiplicity free and that there is a one-to-one correspondence between the labels of the associated U(p, q) and U(n) irreps (Kashiwara and Vergne 1978, King and Wybourne 1985, Quesne 1986). The branching rule is

$$[\rho] \downarrow \sum_{[\boldsymbol{k}][\boldsymbol{k}']}^{(\rho)} \oplus [k_1 \ldots k_p ; k'_1 \ldots k'_q] \times [k_1 \ldots k_p 0 - k'_q \ldots -k'_1] \tag{2.11}$$

where the summation runs over all the partitions $[\boldsymbol{k}] = [k_1 \ldots k_p]$ and $[\boldsymbol{k}'] = [k'_1 \ldots k'_q]$ into p or q non-negative integers, subject to the condition

$$\sum_\alpha k_\alpha - \sum_\beta k'_\beta = \rho. \tag{2.12}$$

Here the indices α and β run from 1 to p and 1 to q, respectively. In equation (2.11), the U(p, q) irreps are positive discrete series ones, specified by their lowest weight $\{k_p + n/2, \ldots, k_1 + n/2; \ k'_q + n/2, \ldots, k'_1 + n/2\}$, that we denote in short by $[k_1 \ldots k_p ; k'_1 \ldots k'_q]$. On the other hand, the U(n) irreps are finite-dimensional ones, characterised by their highest weight $\{k_1 + (p-q)/2, \ldots, k_p + (p-q)/2, (p-q)/2, -k'_q + (p-q)/2, \ldots, -k'_1 + (p-q)/2\}$, that we denote in short by $[k_1 \ldots k_p 0 - k'_q \ldots -k'_1]$; they are of mixed type, i.e. with negative as well as non-negative labels (Flores 1967, King 1970, 1975).

From the above results, it is clear that the operators P_{ij} and $\mathscr{P}_{st}$ generate in $\mathscr{F}_{(n-1)n}$ the U(p, q) $\times$ U(n) representation

$$\sum_{[\boldsymbol{k}][\boldsymbol{k}']} \oplus [k_1 \ldots k_p ; k'_1 \ldots k'_q] \times [k_1 \ldots k_p 0 - k'_q \ldots -k'_1] \tag{2.13}$$

where the summation is now an unrestricted one over all the partitions $[\boldsymbol{k}]$ and $[\boldsymbol{k}']$ into p or q non-negative integers. Let us now consider the subspace $\mathscr{B}_{p,q}$ of U(p, q) lowest weight states, i.e. the set of vectors $|\psi\rangle$ of $\mathscr{F}_{(n-1)n}$ satisfying the conditions

$$E_{ij}|\psi\rangle = 0 \qquad \text{if } 1 \leqslant j < i \leqslant p \text{ or } p+1 \leqslant j < i \leqslant n-1 \tag{2.14}$$

and

$$D_{ij}|\psi\rangle = 0 \qquad \text{if } i = 1, \ldots, p \text{ and } j = p+1, \ldots, n-1. \tag{2.15}$$

From equation (2.13), it is obvious that the operators $\mathscr{P}_{st}$ generate in $\mathscr{B}_{p,q}$ the U(n) representation

$$\sum_{[k][k']} \oplus [k_1 \ldots k_p 0 - k'_q \ldots -k'_1]. \tag{2.16}$$

Any U(n) irrep $[k_1 \ldots k_p 0 - k'_q \ldots -k'_1]$ remains irreducible when restricted to SU(n) and corresponds to the $(n-1)$-row Young diagram

$$[k_1 + k'_1, \ldots, k_p + k'_1, k'_1, k'_1 - k'_q, \ldots, k'_1 - k'_2] \equiv [l_1 l_2 \ldots l_{n-1}]. \tag{2.17}$$

From equation (2.16), it follows that the subrepresentation of SU(n), generated in $\mathscr{B}_{p,q}$ by the operators

$$\mathscr{P}_{st} - (1/n)\delta_{st} \sum_u \mathscr{P}_{uu} \tag{2.18}$$

has the form

$$\begin{aligned}\sum_{[k][k']} &\oplus [k_1 + k'_1, \ldots, k_p + k'_1, k'_1, k'_1 - k'_q, \ldots, k'_1 - k'_2] \\ &= \sum_{[l]} \oplus [l_1 l_2 \ldots l_{n-1}].\end{aligned} \tag{2.19}$$

On the right-hand side of equation (2.19), the summation runs over all the partitions into $n-1$ non-negative integers. Hence it contains exactly one representative from every equivalence class of irreps of SU(n). The $n-2$ subspaces $\mathscr{B}_{p,q}$ of $\mathscr{F}_{(n-1)n}$, for which p and q satisfy equation (2.3), therefore define a family of $n-2$ models of SU(n).

3. Basis states

To find a basis of $\mathscr{B}_{p,q}$, let us solve equations (2.14) and (2.15). Any solution of equation (2.14) is a linear combination of lowest weight states of irreps of the U(p, q) maximal compact subgroup U(p)$\times$U(q). Such lowest weight states can be written as (Moshinsky 1962)

$$\begin{aligned}&\left(\prod_{\alpha=1}^{p} \prod_{s=\alpha}^{n} (\eta_{p-\alpha+1\ldots p, 1\ldots \alpha-1 s})^{n_{\alpha s}}\right) \\ &\qquad \times \left(\prod_{\beta=1}^{q} \prod_{t=1}^{n-\beta+1} (\eta_{n-\beta\ldots n-1, tn-\beta+2\ldots n})^{n'_{\beta t}}\right)|0\rangle\end{aligned} \tag{3.1}$$

where $\eta_{i_1\ldots i_r s_1 \ldots s_r}$ denotes the determinant of order r obtained from rows $i_1, \ldots, i_r$ and columns $s_1, \ldots, s_r$ of the matrix $\|\eta_{is}\|$ and the exponents $n_{\alpha s}$ and $n'_{\beta t}$ may take any non-negative integer values.

Any solution of equation (2.14) can be transformed into a simultaneous solution of equations (2.14) and (2.15) by replacing the standard boson creation operators η_{is} by modified ones $a^\dagger_{is}$, satisfying the traceless conditions (Quesne 1986)

$$\sum_{s=1}^{n} a^\dagger_{is} a^\dagger_{js} = 0 \qquad i = 1, \ldots, p \qquad j = p+1, \ldots, n-1. \tag{3.2}$$

These traceless boson creation operators and the associated annihilation operators a_{is} are the generalisation to U(n) of the O(n) and USp(n) corresponding operators introduced by Lohe and Hurst (1971) and they are defined by

$$a^\dagger_{is}=\begin{cases}\eta_{is}-\sum\limits_{j=1}^{p}\sum\limits_{k,l=p+1}^{n-1}\left(\sum\limits_{t}\eta_{jt}\eta_{kt}\right)\Delta^{-1}_{jk,il}\xi_{ls} & \text{if } i=1,\ldots,p\\ \eta_{is}-\sum\limits_{j,l=1}^{p}\sum\limits_{k=p+1}^{n-1}\left(\sum\limits_{t}\eta_{jt}\eta_{kt}\right)\Delta^{-1}_{jk,li}\xi_{ls} & \text{if } i=p+1,\ldots,n\end{cases} \tag{3.3}$$

and

$$a_{is}=\xi_{is}. \tag{3.4}$$

Here $\mathbf{\Delta}$ is a $pq\times pq$ operator matrix, whose elements are given by

$$\Delta_{ij,kl}=\delta_{ik}E_{lj}+\delta_{lj}E_{ki}\qquad i,k=1,\ldots,p\qquad j,l=p+1,\ldots,n-1 \tag{3.5}$$

and $\mathbf{\Delta}^{-1}$ is its inverse. The operators $a^\dagger_{is}$ and a_{is} are Hermitian conjugates of one another in $\mathscr{B}_{p,q}$ and have the property of transforming any state satisfying equation (2.15) into another state still fulfilling the same equation.

We conclude that a basis of $\mathscr{B}_{p,q}$ is provided by the set of states

$$|\{n_{\alpha s}\},\{n'_{\beta t}\}\rangle =\left(\prod_{\alpha=1}^{p}\prod_{s=\alpha}^{n}(a^\dagger_{p-\alpha+1\ldots p,1\ldots\alpha-1s})^{n_{\alpha s}}\right)\left(\prod_{\beta=1}^{q}\prod_{t=1}^{n-\beta+1}(a^\dagger_{n-\beta\ldots n-1,tn-\beta+2\ldots n})^{n'_{\beta t}}\right)|0\rangle \tag{3.6}$$

where $a^\dagger_{i_1\ldots i_r,s_1\ldots s_r}$ denotes the determinant of order r obtained from rows $i_1,\ldots,i_r$ and columns $s_1,\ldots,s_r$ of the matrix $\|a^\dagger_{is}\|$ and $n_{\alpha s}$, $n'_{\beta t}$ run over all non-negative integers.

4. Application to SU(3)

For SU(3), we obtain a single model space $\mathscr{B}\equiv\mathscr{B}_{1,1}$, which is the subspace of $\mathscr{F}_6$ consisting of all U(1, 1) lowest weight states. In this case, equation (2.14) disappears, and equation (2.15) gives rise to a single condition, namely

$$D_{12}|\psi\rangle=0. \tag{4.1}$$

The traceless boson operators (3.3) and (3.4) are now

$$\begin{aligned}a^\dagger_{1s}&=\eta_{1s}-\left(\sum_t\eta_{1t}\eta_{2t}\right)(E_{11}+E_{22})^{-1}\xi_{2s}\qquad & a_{1s}&=\xi_{1s}\\ a^\dagger_{2s}&=\eta_{2s}-\left(\sum_t\eta_{1t}\eta_{2t}\right)(E_{11}+E_{22})^{-1}\xi_{1s}\qquad & a_{2s}&=\xi_{2s}.\end{aligned} \tag{4.2}$$

Since

$$\left(\sum_t\eta_{1t}\eta_{2t}\right)(E_{11}+E_{22})^{-1}=(E_{11}+E_{22}-2)^{-1}\left(\sum_t\eta_{1t}\eta_{2t}\right) \tag{4.3}$$

we can eliminate the denominators in equation (4.2) and define renormalised traceless boson operators as follows:

$$A^\dagger_{is}=(E_{11}+E_{22}-2)a^\dagger_{is}\qquad A_{is}=a_{is}. \tag{4.4}$$

These new operators are not Hermitian conjugates of one another any more, but they are simple cubic polynomials in η_{is} and ξ_{is}. In terms of them, the basis vectors (3.6) of $\mathscr{B}$ may be written as

$$|n_1n_2n_3n_1'n_2'n_3'\rangle \propto \prod_{s=1}^{3} (A_{1s}^\dagger)^{n_s}(A_{2s}^\dagger)^{n_s'}|0\rangle \tag{4.5}$$

apart from some irrelevant numerical factor.

It is now obvious that such a model of SU(3) coincides with that of Bracken and MacGibbon (1984) if we identify our operators η_{1s}, ξ_{1s}, η_{2s}, ξ_{2s}, $A_{1s}^\dagger$, A_{1s}, $A_{2s}^\dagger$, A_{2s} and $\mathscr{P}_{st}$ with their operators $\bar{\alpha}^s$, α_s, $\bar{\beta}_s$, β^s, $\bar{A}^s$, A_s, $\bar{B}_s$, B^s and A_t^s respectively, and if we set $P=\Sigma_s \eta_{1s}\xi_{1s}$ and $Q=\Sigma_s \eta_{2s}\xi_{2s}$, so that $E_{11}+E_{22}-2=P+Q+1$. As shown by Bracken and MacGibbon, the renormalised traceless boson operators $A_{is}^\dagger$ and A_{is} generate under commutation a representation of the Lie algebra so(6, 2) and, moreover, $\mathscr{B}$ carries an irrep of the corresponding Lie group SO(6, 2). As was stressed by Biedenharn and Flath (1984), the renormalisation plays an essential role in obtaining such results. From the commutation relations of the unrenormalised operators $a_{is}^\dagger$ and a_{is} (Quesne 1986), it is indeed clear that the latter do not generate a closed algebra under commutation.

5. Conclusion

In the present paper, we have obtained a family of $n-2$ models of SU(n) for $n \geqslant 3$, starting with the SO(6, 2) model of SU(3). For $n>3$, such models will be useful if they carry an irrep of some group. Whether this is the case is still an open question, significantly more complex to answer than the corresponding one for $n=3$. One reason for such complexities lies in the renormalisation of the $a_{is}^\dagger$ operators, now containing the operator matrix $\boldsymbol{\Delta}^{-1}$ instead of a function of a number operator as in the $n=3$ case. A further reason is that an attractive feature of the SO(6, 2) model of SU(3) is lost: the models for SU(n), $n>3$, do not admit every vector that can be obtained from the vacuum state by application of the traceless boson creation operators; consequently the counterpart of so(6, 2) for $n>3$, if it does exist, will not be simply generated by commuting the renormalised traceless boson operators.

As a final comment, we would like to stress that the SU(2, 1) model of SU(2) (Bracken and MacGibbon 1984, Bracken 1984, Van der Jeugt 1985) does not belong to the family of models considered in the present paper. Hence Bracken's suggestion (1984) of considering the construction of a model for SU(4) as equivalent to finding the next Lie algebra in the sequence su(2, 1), so(6, 2), ... may be meaningless. The present paper shows that any sequence, if it exists, should most probably start with so(6, 2).

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